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Dense Hadronic Matter in Holographic QCD

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Abstract

We provide a method to study hadronic matter at finite density in the context of the Sakai-Sugimoto model. We introduce the baryon chemical potential through the external $U(1)_v$ in the induced (DBI plus CS) action on the D8-probe-brane, where baryons are skyrmions. Vector dominance is manifest at finite density. We derive the baryon density effect on the energy density, the dispersion relations of pion and vector mesons at large N_c. The energy density asymptotes to a constant at large density suggesting that dense matter at large N_c freezes, with the pion velocity dropping to zero. Holographic dense matter enforces exactly the tenets of vector dominance, and screens efficiently vector mesons. At the freezing point the $\rho - \pi\pi$ coupling vanishes with a finite rho mass of about 20% its vacuum value.

1 Introduction

Recently there has been much interest [1, 2] in the AdS/CFT approach [3] to study the non-perturbative aspects of gauge theories at large N_c and strong coupling with even applications to heavy ion collision experiments such as jet quenching [4]. On the other hand, dense QCD is of much relevance to compact stars and current heavy ion collisions. This subject has been intensely studied in the past decade following the observation that at asymptotic densities QCD matter may turn to a color superconductor due to asymptotic freedom [5]. At very large N_c there is evidence that the Overhauser effect takes over with the formation of density waves [6]. Indeed, at weak coupling the high degeneracy of the Fermi surface causes quark-quark (antiquark-antiquark) pairing of the BCS kind with different color-flavor arrangements, while at large number of colors quark-antiquark pairing of the Overhauser kind is favored. In strong coupling, first principle calculations are elusive and the two pairings may compete.

To address the issue of strongly coupled QCD at large N_c in dense matter we explore in this paper the holographic principle. In recent years, the gauge/string duality has provided a framework for addressing a number of problems in strong coupling where little is known from first principles in the continuum. To do so, we consider a model for holographic QCD recently discussed by Sakai and Sugimoto. In the limit $N_f \ll N_c$, chiral symmetry in QCD is generated by immersing N_f D8-$\overline{\text{D8}}$ into a D4 background in 5 dimensions where supersymmetry is broken by compactification (Kaluza-Klein mechanism with M_{KK} scale). The induced DBI action on D8 yields a 4 dimensional effective theory of pions and infinitly many vector mesons where the M_{KK} scale plays the role of an upper cutoff (the analogue of the chiral scale $4\pi\, f_\pi$). At large N_c baryons are Skyrmions.

In this paper, we extend the analysis by Sakai and Sugimoto [2] to finite baryon density. Although there have been many papers [7] on finite chemical potential in the holographic approach, all involve the R-charge or isospin instead of fermion number. The difficulty for the fermion number is that the $U(1)$ charge for R-symmetry is already dual to the $U(1)$ charge of AdS Reisner-Nordstrom Black hole charge. There is no additional $U(1)$ in bulk to modify the geometry of the AdS black hole. In this paper we follow the more traditional route of introducing the chemical potential via the nonperturbative induced DBI plus CS action through the external $U(1)_V$ source in Sakai and Sugimoto chiral model. Specifically, we introduce the baryon chemical potential as $\mathcal{V}_0 = -i\mu_B/N_c$ where $\mathcal{V}_\mu$ is the external vector field in the induced DBI plus CS action. While $\mathcal{V}_0$ drops in the DBI part it contributes to the CS part through the WZW as it should. A redefinition of the vector fields in the Sakai-Sugimoto model necessary to enforce vector dominance, causes a reshuffling of $\mathcal{V}_0$ from the

CS to the DBI action as we detail below, with important phenomenological consequences.

In section 2 we review the emergence of the induced effective action on the probe D8 brane from the 5 dimensional D4-brane background, and we detail the contributions of the DBI action to fourth order in the vector fields, as well as the Chern-Simons part. The expanded effective action enforces total vector dominance. In section 3 we introduce the baryon chemical potential and details the nature of the effective expansion at finite density. In section 4, we extract the baryon number density, the pressure and the energy density of holographic dense matter. In section 5, we discuss the pion and leading vector meson parameters and interactions in holographic dense matter. Our conclusions and suggestions are summarized in section 6.

2 Duality and Branes

The *generalized holographic principle* states that at large N_c maximally supersymmetric and conformal gauge theories in flat D-dimensions are dual to Superstring/Mtheory on pertinent AdS spaces, i.e. for $D = 4$ it is $AdS_5 \times S^5$ and for $D = 6$ it is $AdS_7 \times S^4$.

To model QCD in $D = 4$ dimensions, we start from $D = 6$ superconformal theory, and compactify it twice. First through S^1 with radius R_1 leading to $D = 5$ SUSY theory. Second through another S^1 of radius $R_2 \gg R_1$ with fermion antiperiodic boundary conditions to break SUSY. Below the cutoff scale $M_{KK} = 1/R_2$, the D=5 broken SUSY theory behaves as a D=4 gauge theory.

2.1 D4/D8 Branes

Doubly compactified D=6 is the boundary of AdS_7 in which the M-theory reduces to type IIA string theory. As a result M5 branes wrapping around the first S^1 with radius R^1 transmute to D4-branes. N_c copies of these branes yield near extremal black-hole background.

The metric, dilaton ϕ and the RR three-form field C_3 in a D4-brane background are given by

$$ds^2 = \left(\frac{U}{R}\right)^{3/2} \left(\eta_{\mu\nu} dx^\mu dx^\nu + f(U) d\tau^2\right) + \left(\frac{R}{U}\right)^{3/2} \left(\frac{dU^2}{f(U)} + U^2 d\Omega_4^2\right),$$

$$e^\phi = g_s \left(\frac{U}{R}\right)^{3/4}, \quad F_4 \equiv dC_3 = \frac{2\pi N_c}{V_4} \epsilon_4 , \quad f(U) \equiv 1 - \frac{U_{\rm KK}^3}{U^3} . \tag{1}$$

Here $\mu = 0, 1, 2, 3$ and τ is the compact variable on S^1. $U > U_{KK}$ is the radial coordinate along 56789, ϵ_4 the 4-form and $V_4 = 8\pi^2/3$ the volume of a unit S^4 surrounding the D-4 brane. $R^3 \equiv \pi g_s N_c l_s^3$, where g_s and l_s are the string coupling and length respectively. Again, this background represents N_c D-4 branes wrapped on S^1.

Let us consider N_f $D8$-probe-branes in this background, which may be described by $\tau(U)$. We choose a specific configuration such that τ is constant($\tau = \frac{\pi}{4M_{KK}}$). It corresponds to the maximal asymptotic separation between $D8$ and $\bar{D}8$. with a new variable z, instead of U, defined by the relation,

$$U = (U_{KK}^3 + U_{KK} z^2)^{\frac{1}{3}} \equiv U_z \tag{2}$$

the induced metric on D8 is

$$\begin{aligned} ds_{D8}^2 &= g_{MN} dx^M dx^N \\ &= \left(\frac{U_z}{R}\right)^{3/2} \eta_{\mu\nu} dx^\mu dx^\nu + \frac{4}{9}\left(\frac{R}{U_z}\right)^{3/2} \frac{U_{KK}}{U_z} dz^2 + \left(\frac{R}{U_z}\right)^{3/2} U_z^2 d\Omega_4^2 \ , \end{aligned} \tag{3}$$

where

$$M(N) \ = \ \{\mu(\nu)(0,1,2,3), z(4), \alpha(5,6,7,8)\} \tag{4}$$

2.2 DBI and CS Action on D8

Consider the $U(N_f)$ gauge field A_M on the probe D8-brane configuration. The effective action is 9-dimensional and reads

$$S_{D8} = S_{DBI} + S_{CS} \ ,$$

$$S_{DBI} = -T \int d^9x \, \mathrm{tr}\left(e^{-\phi}\sqrt{-\det(g_{MN} + 2\pi\alpha' F_{MN})}\right) \tag{5}$$

$$S_{CS} = \frac{1}{48\pi^3}\int_{D8} C_3 \, \mathrm{tr}\, F^3 = \frac{1}{48\pi^3}\int_{D8} F_4 \, \omega_5(A) \ , \tag{6}$$

where $e^{-\phi} = g_s \left(\frac{U_z}{R}\right)^{-\frac{3}{4}}$, $F_4 = dC_3$ is the RR 4-form field strength and $\omega_5(A)$ is the Chern-Simons 5-form,

$$\omega_5(A) = \mathrm{tr}\left(AF^2 - \frac{1}{2}A^3 F + \frac{1}{10}A^5\right) \ , \tag{7}$$

which satisfies $d\omega_5 = \mathrm{tr}\, F^3$.

Assuming that $A_\alpha = 0$ and A_μ and A_z are independent of the coordinates on the S^4, the induced DBI action becomes 5-dimensional

$$
\begin{aligned}
S_{DBI} &= -T \int d^9x \, \mathrm{tr} \left(e^{-\phi} \sqrt{-\det(g_{MN} + 2\pi\alpha' F_{MN})} \right) && (8) \\
&= -\widetilde{T} \int d^4x dz \, U_z^2 \\
&\quad \mathrm{tr} \sqrt{1 + (2\pi\alpha')^2 \frac{R^3}{2U_z^3} \eta^{\mu\nu}\eta^{\rho\sigma} F_{\mu\rho} F_{\nu\sigma} + (2\pi\alpha')^2 \frac{9}{4} \frac{U_z}{U_{\mathrm{KK}}} \eta^{\mu\nu} F_{\mu z} F_{\nu z} + [F^4] + [F^5]} \ , && (9)
\end{aligned}
$$

where $\widetilde{T} = \frac{N_c}{216\pi^5} \frac{M_{KK}}{\alpha'^3}$. $[F^4]$ and $[F^5]$ are short for structures of the type

$$
[F^4] \sim F_{\mu z}^2 F_{\mu\nu}^2 + F_{\mu\nu}^4 \tag{10}
$$

$$
[F^5] \sim F_{\mu z}^2 F_{\mu\nu}^3 \tag{11}
$$

The Chern-Simons term is

$$
S_{CS}^{D8} = \frac{N_c}{24\pi^2} \int_{M^4 \times \mathbb{R}} \omega_5(A) \ , \tag{12}
$$

with the normalization $\frac{1}{2\pi} \int_{S^4} F_4 = N_c$, and $M^4 \times \mathbb{R}$ is the five-dimensional plane parameterized by x^μ and z.

2.3 Effective Action in 4-dimension

In this part we review how the 5-dimendional DBI action yields the 4-dimensional effective action. Essentially, the 5-dimensional induced action with compact S^1 is 4-dimensional for all excitations with wavelengths larger than the compactification radius of the order of $1/M_{KK}$. Throughout and for zero baryon density, our discussion parallels the original discussion by Sakai and Sugimoto [2]. We quote it for notation and completeness, and refer to their work for further details.

The leading terms in the $1/\lambda \approx 1/M_{KK}$ expansion of the DBI action is

$$
S_{\mathrm{D8}}^{\mathrm{DBI}} = \kappa \int d^4x dZ \ \mathrm{tr} \left[\frac{1}{2} K^{-1/3} F_{\mu\nu}^2 + K F_{\mu z}^2 \right] \ , \tag{13}
$$

where

$$\kappa \equiv \widetilde{T}(2\pi\alpha')^2 R^3 = \frac{\lambda N_c}{108\pi^3} \ , \quad Z \equiv \frac{z}{U_{KK}} \ , \quad K \equiv 1 + Z^2 \ . \tag{14}$$

In order to extract four-dimensional meson fields out of the five dimensional gauge field, we expand the gauge field as

$$\begin{aligned} A_\mu(x^\mu, z) &= \sum_{n=1}^{\infty} B_\mu^{(n)}(x^\mu)\psi_n(z) \ , \quad (15)\\ A_z(x^\mu, z) &= \varphi^{(0)}(x^\mu)\phi_0(z) + \sum_{n=1}^{\infty} \varphi^{(n)}(x^\mu)\phi_n(z) \ , \quad (16) \end{aligned}$$

We choose the functions $\psi_n(z)$ to be the eigenfunctions satisfying the self-adjoint differential equation

$$-K^{1/3}\,\partial_Z\left(K\,\partial_Z\psi_n\right) = \lambda_n\psi_n \tag{17}$$

where λ_n is the eigenvalue and $\{\psi_n\}$ is a complete set. the normalization is fixed by the Laplacian in (17)

$$\kappa \int dZ\, K^{-1/3}\psi_n\psi_m \;=\; \delta_{nm} \ . \tag{18}$$

This implies

$$\kappa \int dZ\, K\partial_Z\psi_n\partial_Z\psi_m \;=\; \lambda_n\delta_{nm} \tag{19}$$

The $\phi_n(Z)$ are chosen such that

$$\begin{aligned} \phi_n(Z) &= (m_n U_{KK})^{-1}\partial_Z\psi_n(Z) \quad (n \geq 1)\\ \phi_0(Z) &= \frac{1}{\sqrt{\pi\kappa}M_{kk}U_{KK}K} \end{aligned} \tag{20}$$

satisfying the normalization condition:

$$(\phi_m, \phi_n) \equiv \frac{9}{4}\widetilde{T}(2\pi\alpha')^2 U_{\rm KK}^3 \int dZ\, K\,\phi_m\phi_n = \delta_{mn} \ , \tag{21}$$

which is compatible with (17) and (18).

Inserting (15) and (16) into (13) and using the orthonomality of ψ_n and ϕ_n, yield

$$S_{DBI} = \int d^4x \ \mathrm{tr} \left[(\partial_\mu \varphi^{(0)})^2 + \sum_{n=1}^{\infty} \left(\frac{1}{2} (\partial_\mu B_\nu^{(n)} - \partial_\nu B_\mu^{(n)})^2 + \lambda_n M_{KK}^2 (B_\mu^{(n)} - \lambda_n^{-1/2} \partial_\mu \varphi^{(n)})^2 \right) \right] + (\text{interaction terms}) \ . \tag{22}$$

In the expansion (15) and (16), we have implicitly assumed that the gauge fields are zero asymptotically, i.e. $A_M(x^\mu, z) \to 0$ as $z \to \pm\infty$. The residual gauge transformation that does not break this condition is obtained by a gauge function $g(x^\mu, z)$ that asymptotes a constant $g(x^\mu, z) \to g_\pm$ at $z \pm \infty$. We interpret (g_+, g_-) as elements of the chiral symmetry group $U(N_f)_L \times U(N_f)_R$ in QCD with N_f massless flavors.

2.3.1 External photons

By weakly gauging the $U(N_f)_L \times U(N_f)_R$ chiral symmetry, we may introduce the external gauge fields $(A_{L\mu}, A_{R\mu})$. For the interaction between mesons and photon A_μ^{em}, we may choose

$$A_{L\mu} = A_{R\mu} = eQA_\mu^{\mathrm{em}} \ , \tag{23}$$

where e is the electromagnetic coupling constant and Q is the electric matrix-valued charge

$$Q = \frac{1}{3} \begin{pmatrix} 2 & & \\ & -1 & \\ & & -1 \end{pmatrix} \ , \tag{24}$$

for $N_f = 3$. To insert the external-source gauge fields, we impose the asymptotic values of the gauge field A_μ on the D8-probe-brane as

$$\lim_{z \to +\infty} A_\mu(x^\mu, z) = A_{L\mu}(x^\mu) \ , \qquad \lim_{z \to -\infty} A_\mu(x^\mu, z) = A_{R\mu}(x^\mu) \ . \tag{25}$$

This is implemented by modifying the mode expansion (15) as

$$A_\mu(x^\mu, z) = A_{L\mu}(x^\mu)\psi_+(z) + A_{R\mu}(x^\mu)\psi_-(z) + \sum_{n=1}^{\infty} B_\mu^{(n)}(x^\mu)\psi_n(z) \ , \tag{26}$$

where $\psi_\pm(z)$ are defined as

$$\psi_\pm(z) \equiv \frac{1}{2}(1 \pm \psi_0(z)) \ , \quad \psi_0(z) \equiv \frac{2}{\pi} \arctan z \ , \tag{27}$$

which are the non-normalizable zero modes of (17) satisfying $\partial_z \psi_\pm(z) \propto \phi_0(z)$.

2.3.2 $A_z = 0$ gauge

The $A_z = 0$ gauge can be achieved by applying the gauge transformation $A_M \to g A_M g^{-1} + g\partial_M g^{-1}$ with the gauge function

$$g^{-1}(x^\mu, z) = P \exp \left\{ -\int_0^z dz' \, A_z(x^\mu, z') \right\} \ . \tag{28}$$

The asymptotic values of (25) change to

$$\lim_{z\to+\infty} A_\mu(x^\mu, z) = A_{L\mu}^{\xi_+}(x^\mu) \ , \qquad \lim_{z\to-\infty} A_\mu(x^\mu, z) = A_{R\mu}^{\xi_-}(x^\mu) \ , \tag{29}$$

and

$$A_{L\mu}^{\xi_+}(x^\mu) \ \equiv \ \xi_+(x^\mu) A_{L\mu}(x^\mu) \xi_+^{-1}(x^\mu) + \xi_+(x^\mu)\partial_\mu \xi_+^{-1}(x^\mu) \ , \tag{30}$$

$$A_{R\mu}^{\xi_-}(x^\mu) \ \equiv \ \xi_-(x^\mu) A_{R\mu}(x^\mu) \xi_-^{-1}(x^\mu) + \xi_-(x^\mu)\partial_\mu \xi_-^{-1}(x^\mu) \ . \tag{31}$$

where $\xi_\pm(x^\mu) \equiv \lim_{z\to\pm\infty} g(x^\mu, z)$. The gauge field in the $A_z = 0$ gauge can be expanded as

$$A_\mu(x^\mu, z) \ = \ A_{L\mu}^{\xi_+}(x^\mu)\psi_+(z) + A_{R\mu}^{\xi_-}(x^\mu)\psi_-(z) + \sum_{n=1}^{\infty} B_\mu^{(n)}(x^\mu)\psi_n(z) \ . \tag{32}$$

The residual gauge symmetry in the $A_z = 0$ gauge is given by the z-independent gauge transformation. The residual gauge symmetry $h(x^\mu) \in U(N_f)$ and the weakly gauged chiral symmetry $(g_+(x^\mu), g_-(x^\mu)) \in U(N_f)_L \times U(N_f)_R$ act on these fields as

$$A_{L\mu} \ \to \ g_+ A_{L\mu} g_+^{-1} + g_+ \partial_\mu g_+^{-1} \ , \tag{33}$$

$$A_{R\mu} \ \to \ g_- A_{R\mu} g_-^{-1} + g_- \partial_\mu g_-^{-1} \ , \tag{34}$$

$$\xi_\pm \ \to \ h \, \xi_\pm \, g_\pm^{-1} \ , \tag{35}$$

$$B_\mu^{(n)} \ \to \ h \, B_\mu^{(n)} \, h^{-1} \ . \tag{36}$$

Here $\xi_\pm(x^\mu)$ are interpreted as the $U(N_f)$ valued fields $\xi_{L,R}(x^\mu)$ which carry the pion degrees of freedom in the hidden local symmetry approach. Actually the transformation property (35) is the same as that for $\xi_{L,R}(x^\mu)$ if we interpret $h(x^\mu) \in U(N_f)$ as the hidden local symmetry. They are related to the $U(N_f)$ valued pion field $U(x^\mu)$ in the chiral Lagrangian by

$$\xi_+^{-1}(x^\mu)\xi_-(x^\mu) = U(x^\mu) \equiv e^{2i\Pi(x^\mu)/f_\pi} \ . \tag{37}$$

The pion field $\Pi(x^\mu)$ is identical to $\varphi^{(0)}(x^\mu)$ in (16) in leading order. [1]

[1] Here the pion field $\Pi(x^\mu)$ is a Hermitian matrix, while $\varphi^{(0)}(x^\mu)$ and the vector meson fields $B_\mu^{(n)}(x^\mu)$ are anti-Hermitian.

If we choose the gauge

$$\xi_-(x^\mu) = 1, \quad \xi_+^{-1}(x^\mu) = U(x^\mu) = e^{2i\Pi(x^\mu)/f_\pi} \tag{38}$$

then the gauge field is

$$\begin{aligned} A_\mu(x^\mu, z) &= U^{-1}(x^\mu) A_{L\mu}(x^\mu) U(x^\mu) \psi_+(z) + A_{R\mu}(x^\mu)\psi_-(z) \\ &+ U^{-1}(x^\mu)\partial_\mu U(x^\mu)\psi_+(z) + \sum_{n\geq 1} B_\mu^{(n)}(x^\mu)\psi_n(z) \end{aligned} \tag{39}$$

An alternative gauge is also

$$\xi_+^{-1}(x^\mu) = \xi_-(x^\mu) = e^{i\Pi(x^\mu)/f_\pi} \ . \tag{40}$$

In this gauge, the gauge potential in (32) can be expanded up to quadratic order in fields as

$$\begin{aligned} A_\mu &= \left(\mathcal{V}_\mu + \frac{1}{2f_\pi^2}[\,\Pi, \partial_\mu \Pi\,] - \frac{i}{f_\pi}[\,\Pi, \mathcal{A}_\mu\,] \right) + \left(\mathcal{A}_\mu + \frac{i}{f_\pi}\partial_\mu \Pi - \frac{i}{f_\pi}[\,\Pi, \mathcal{V}_\mu\,] \right) \psi_0 + \\ &+ \sum_{n=1}^{\infty} v_\mu^n \, \psi_{2n-1} + \sum_{n=1}^{\infty} a_\mu^n \, \psi_{2n} + \cdots \ , \end{aligned} \tag{41}$$

with

$$\mathcal{V}_\mu \equiv \frac{1}{2}(A_{L\mu} + A_{R\mu}) \ , \quad \mathcal{A}_\mu \equiv \frac{1}{2}(A_{L\mu} - A_{R\mu}) \ , \quad v_\mu^n \equiv B_\mu^{(2n-1)} \ , \quad a_\mu^n \equiv B_\mu^{(2n)} \ . \tag{42}$$

2.3.3 Skyrme action and WZW term

It is interesting to note with Sakai and Sugimoto [2] that the induced effective action expanded to fourth order in the gradients yields exactly a Skyrme-like action

$$S_{DBI}\Big|_{v_\mu^n = a_\mu^n = \mathcal{V}_\mu = \mathcal{A}_\mu = 0} = \int d^4x \left(\frac{f_\pi^2}{4} \,\mathrm{tr}\left(U^{-1}\partial_\mu U\right)^2 + \frac{1}{32 e_S^2} \,\mathrm{tr}\left[U^{-1}\partial_\mu U, U^{-1}\partial_\nu U\right]^2 \right) \ , \tag{43}$$

where the pion decay constant f_π and the dimensionless parameter e_S are fixed

$$f_\pi^2 \equiv \frac{4}{\pi}\kappa = \frac{1}{27\pi^4}\lambda N_c \ , \tag{44}$$

$$e_S^{-2} \equiv \kappa \int dz\, K^{-1/3}(1 - \psi_0^2)^2 \ . \tag{45}$$

Also in the $A_z = 0$ gauge, the Chern-Simons term is generic

$$
\begin{aligned}
S^{\rm CS}_{\rm D8} &= -\frac{N_c}{24\pi^2}\int_{M^4}\big(\alpha_4(d\xi_+^{-1}\xi_+, A_L) - \alpha_4(d\xi_-^{-1}\xi_-, A_R)\big) + \\
&\quad + \frac{N_c}{24\pi^2}\int_{M^4\times\mathbb{R}}\left(\omega_5(A) - \frac{1}{10}\,{\rm tr}(gdg^{-1})^5\right) \ , \qquad (46)
\end{aligned}
$$

where g is the gauge function given in (28) and the 4-form α_4 is

$$
\alpha_4(V, A) \equiv -\frac{1}{2}\,{\rm tr}\left(V(AdA + dAA + A^3) - \frac{1}{2}VAVA - V^3A\right) \ . \qquad (47)
$$

As expected (46) yields the WZW term in QCD with gauged pion fields. Indeed, if we omit the vector meson fields $B_\mu^{(n)}$, we have

$$
S_{CS}\Big|_{v_\mu^n = a_\mu^n = 0} = -\frac{N_c}{48\pi^2}\int_{M^4} Z - \frac{N_c}{240\pi^2}\int_{M^4\times\mathbb{R}}{\rm tr}(gdg^{-1})^5 \ , \qquad (48)
$$

where

$$
\begin{aligned}
Z &= {\rm tr}[(A_RdA_R + dA_RA_R + A_R^3)(U^{-1}A_LU + U^{-1}dU) - {\rm p.c.}] + \\
&\quad + {\rm tr}[dA_RdU^{-1}A_LU - {\rm p.c.}] + {\rm tr}[A_R(dU^{-1}U)^3 - {\rm p.c.}] + \\
&\quad + \frac{1}{2}\,{\rm tr}[(A_RdU^{-1}U)^2 - {\rm p.c.}] + {\rm tr}[UA_RU^{-1}A_LdUdU^{-1} - {\rm p.c.}] - \\
&\quad - {\rm tr}[A_RdU^{-1}UA_RU^{-1}A_LU - {\rm p.c.}] + \frac{1}{2}\,{\rm tr}[(A_RU^{-1}A_LU)^2] \ . \qquad (49)
\end{aligned}
$$

Here "p.c." represents the terms obtained by exchanging $A_L \leftrightarrow A_R,\ U \leftrightarrow U^{-1}$.

2.3.4 Vector meson dominance

The induced DBI action (13) carries exact vector dominance. Indeed, by diagonalizing the kinetic terms of the vector meson fields, vector meson dominance emerges from the underlying DBI action naturally. To see that, we recall that the gauge fields are

$$
\begin{aligned}
A_\mu &= \mathcal{V}_\mu + \mathcal{A}_\mu\psi_0 + \sum_{n=1}^{\infty} v_\mu^n\,\psi_{2n-1} + \sum_{n=1}^{\infty} a_\mu^n\,\psi_{2n} \ , \qquad (50) \\
A_z &= -i\,\Pi\,\phi_0 \ , \qquad (51)
\end{aligned}
$$

and involve both the external vector sources $\mathcal{V}, \mathcal{A}$ and the dynamical vector fields V, a. To diagonalize the kinetic terms of the dynamical vector fields require the introduction of the physical tilde vector fields $\tilde{v}, \tilde{a}$ which are

$$
\begin{aligned}
\widetilde{v}^n_\mu &\equiv v^n_\mu + a_{\mathcal{V}v^n}\mathcal{V}_\mu \ , && (52)\\
\widetilde{a}^n_\mu &\equiv a^n_\mu + a_{\mathcal{A}a^n}\mathcal{A}_\mu \ \ , && (53)
\end{aligned}
$$

in terms of which the gauge fields now read

$$
A_\mu \ = \ \mathcal{V}_\mu\,\psi_v + \mathcal{A}_\mu\psi_a + \sum_{n=1}^{\infty}\widetilde{v}^n_\mu\,\psi_{2n-1} + \sum_{n=1}^{\infty}\widetilde{a}^n_\mu\,\psi_{2n} \ , \tag{54}
$$

with

$$
\begin{aligned}
&\psi_v \equiv 1 - \sum_{n=1}^{\infty} a_{\mathcal{V}v^n}\psi_{2n-1} \ , \quad \psi_a \equiv \psi_0 - \sum_{n=1}^{\infty} a_{\mathcal{A}a^n}\psi_{2n} \ .\\
&a_{\mathcal{V}v^n} \equiv \alpha^n \equiv \kappa\int dz\, K^{-1/3}\psi_{2n-1} \ , \quad a_{\mathcal{A}a^n} \equiv \kappa\int dz\, K^{-1/3}\psi_{2n}\psi_0 \ ,
\end{aligned} \tag{55}
$$

The DBI action (13) in terms of the physical vector fields to fourth order is

$$
\begin{aligned}
&\kappa\int dz\ \mathrm{tr}\left[\frac{1}{2}K^{-1/3}F^2_{\mu\nu}\right]\\
= \ & \mathrm{tr}\Bigg[\ \frac{1}{2e^2}\left((F^{A_L}_{\mu\nu})^2 + (F^{A_R}_{\mu\nu})^2\right) +\\
&+ \frac{1}{2}(\partial_\mu\widetilde{v}^n_\nu - \partial_\nu\widetilde{v}^n_\mu)^2 + \frac{1}{2}(\partial_\mu\widetilde{a}^n_\nu - \partial_\nu\widetilde{a}^n_\mu)^2 +\\
&+ (\partial_\mu\widetilde{v}^n_\nu - \partial_\nu\widetilde{v}^n_\mu)([\widetilde{v}^{p\mu},\widetilde{v}^{q\nu}]\,g_{v^nv^pv^q} + [\widetilde{a}^{p\mu},\widetilde{a}^{q\nu}]\,g_{v^na^pa^q}) +\\
&+ (\partial_\mu\widetilde{a}^n_\nu - \partial_\nu\widetilde{a}^n_\mu)([\widetilde{v}^{p\mu},\widetilde{a}^{q\nu}] - [\widetilde{v}^{q\nu},\widetilde{a}^{p\mu}])\,g_{v^pa^na^q} +\\
&+ \frac{1}{2}[\widetilde{v}^m_\mu,\widetilde{v}^n_\nu][\widetilde{v}^{p\mu},\widetilde{v}^{q\nu}]\,g_{v^mv^nv^pv^q} + \frac{1}{2}[\widetilde{a}^m_\mu,\widetilde{a}^n_\nu][\widetilde{a}^{p\mu},\widetilde{a}^{q\nu}]\,g_{a^ma^na^pa^q} +\\
&+ \left([\widetilde{v}^m_\mu,\widetilde{v}^n_\nu][\widetilde{a}^{p\mu},\widetilde{a}^{q\nu}] + [\widetilde{v}^m_\mu,\widetilde{a}^p_\nu][\widetilde{v}^{n\mu},\widetilde{a}^{q\nu}] - [\widetilde{v}^m_\mu,\widetilde{a}^p_\nu][\widetilde{v}^{n\nu},\widetilde{a}^{q\mu}]\right)\,g_{v^mv^na^pa^q}\Bigg] \ .
\end{aligned} \tag{56}
$$

Note that all the couplings between the external gauge fields (A_L, A_R) and the vector meson fields $(\widetilde{v}^n, \widetilde{a}^n)$ vanish in the first term of the effective action (13). The second term in the effective action (13)

$$
\begin{aligned}
& \kappa \int dz \ \mathrm{tr}\left[K F_{z\nu}^2\right] \\
= & \ \mathrm{tr}\Big[m_{v^n}^2 (\widetilde{v}_\mu^n - a_{\mathcal{V}v^n}\mathcal{V}_\mu)^2 + m_{a^n}^2 (\widetilde{a}_\mu^n - a_{\mathcal{A}a^n}\mathcal{A}_\mu)^2 + (i\partial_\mu \Pi + f_\pi \mathcal{A}_\mu)^2 + \\
& + 2i g_{a^m v^n \pi}\, \widetilde{a}_\mu^m [\Pi, \widetilde{v}^{n\mu}] - 2 g_{v^n \pi\pi}\, \widetilde{v}_\mu^n [\Pi, \partial^\mu \Pi] - \\
& - c_{a^n a^m} [\Pi, \widetilde{a}_\mu^n][\Pi, \widetilde{a}^{n\mu}] - c_{v^n v^m} [\Pi, \widetilde{v}_\mu^n][\Pi, \widetilde{v}^{n\mu}] \Big] \ ,
\end{aligned}
\tag{57}
$$

The mesons couple to the external gauge fields only through $\widetilde{v}^n \to \mathcal{V}$ and $\widetilde{a}^n \to \mathcal{A}$ transitions in (57).

Finally, inserting (50) and (51) into the Chern-Simons action gives

$$
\begin{aligned}
S_{\mathrm{D8}}^{\mathrm{CS}} = & -\frac{N_c}{4\pi^2}\frac{i}{f_\pi}\int_{M^4} \mathrm{tr}\Big[\Pi \, dB^n dB^m \, c_{nm} + \\
& + \Pi \, (dB^m B^n B^p + B^m B^n dB^p)\, c_{mnp} + \Pi \, B^m B^n B^p B^q \, c_{mnpq} \Big] + \\
& + \frac{N_c}{24\pi^2} \int_{M^4} \mathrm{tr}\Big[B^m B^n dB^p \, d_{mn|p} - \frac{3}{2} B^m B^n B^p B^q \, d_{mnp|q} \Big] \ ,
\end{aligned}
\tag{58}
$$

where $B^{2n-1} \equiv \widetilde{v}^n$, $B^{2n} \equiv \widetilde{a}^n$ It shows complete vector meson dominace in the WZW term. The terms with two or more pion fields are absent.

3 Induced action at finite density

For simplicity we work in Euclidean space, in which case the DBI action reads

$$
\begin{aligned}
S_{DBI}^E = & \ \widetilde{T} \int d^4 x^E dz \, U_z^2 \\
& \mathrm{tr} \sqrt{1 + (2\pi\alpha')^2 \frac{R^3}{2U_z^3} F_{\mu\nu}^E F_{\mu\nu}^E + (2\pi\alpha')^2 \frac{9}{4}\frac{U_z}{U_{\mathrm{KK}}} F_{\mu z}^E F_{\mu z}^E + [(F^E)^4] + [(F^E)^5]} \ ,
\end{aligned}
\tag{59}
$$

following prescription.

$$
\begin{aligned}
x^0 & \to -i\tau \ , \\
A_0 & \to i A_0^E \ , \\
S^E & = -iS \ .
\end{aligned}
\tag{60}
$$

From now on we omit the superscript E. From the arguments to follow, we will see that due to vector dominance no *matter term* is generated through the Chern-Simons part. In a way this is expected; the anomaly in the Chern-Simons form is an ultraviolet effect that after regularization shows up in the infrared. Since matter dwells mostly in the infrared, the scale decoupling insures the insensitivity of the Chern-Simons terms therefore of the anomaly. This result is usually referred to as the non-renormalization theorem.

The density will be introduced through the boundary and via the external vector field $\mathcal{V}$ through

$$\mathcal{V}_\mu = -i\,\mu\,\delta_{\mu 0}\,1_{N_f\times N_f}, \qquad \mathcal{A}_\mu = 0\ . \tag{61}$$

The master gauge field now reads from (50)

$$A_\mu \;=\; -i\mu\delta_{\mu 0}\,(1-\sum_{n=1}^{\infty}\alpha^n\psi_{2n-1}) + \sum_{n=1}^{\infty}\widetilde{v}_\mu^n\,\psi_{2n-1} + \sum_{n=1}^{\infty}a_\mu^n\,\psi_{2n}\ , \tag{62}$$

For calculational convenience below, we split $\widetilde{v}_0^n$ in two parts: a constant $U(1)$ part $(-i\widetilde{v}_{0C}^n)$ and the rest $(\overline{v_{0C}^n})$,

$$\begin{aligned}
\widetilde{v}_0^n &= -iv_0^n - i\alpha^n\mu = -iv_{0C}^n + \overline{v_{0C}^n} - i\alpha^n\mu \\
&= \underbrace{-iv_{0C}^n - i\alpha^n\mu}_{=:-i\widetilde{v}_{0C}^n} + \overline{v_{0C}^n}, \\
\widetilde{v}_i^n &\equiv v_i^n
\end{aligned}$$

where $-iv_{0C}^n$ and $\overline{v_{0C}^n}$ are the constant $U(1)$ part and the rest part of v_0^n respectively. Thus

$$\begin{aligned}
A_0 &= -i\mu + \sum_{n=1}^{\infty} i(\mu\alpha^n - \widetilde{v}_{0C}^n)\psi_{2n-1} + \sum_{n=1}^{\infty}\overline{v_{0C}^n}\,\psi_{2n-1} + \sum_{n=1}^{\infty}a_0^n\,\psi_{2n}\ , \\
A_i &= \sum_{n=1}^{\infty}v_i^n\,\psi_{2n-1} + \sum_{n=1}^{\infty}a_i^n\,\psi_{2n}\ ,
\end{aligned} \tag{63}$$

Defining,

$$\begin{aligned}
X^{2n-1} &:= \mu\alpha^n - \widetilde{v}_{0C}^n \\
X^{2n} &:= 0 \\
\bar{B}_0^{2n-1} &:= \overline{v_{0C}^n} \quad \bar{B}_i^{2n-1} := v_i^n \\
\bar{B}_\mu^{2n} &:= a_\mu^n
\end{aligned} \tag{64}$$

leads

$$
\begin{aligned}
A_\mu(x^\mu, z) &= -i\mu\delta_{\mu 0} + \sum_{n=1}^{\infty}\{iX^n\delta_{\mu 0} + \bar{B}^n_\mu(x^\mu)\}\psi_n(z) \ , \\
A_z(x^\mu, z) &= -i\Pi(x)\phi_0(z) \ . \qquad (65)
\end{aligned}
$$

The corresponding field strengths are

$$
F_{\mu\nu} = \sum_{n=1}^{\infty}(\partial_\mu \bar{B}^n_\nu - \partial_\nu \bar{B}^n_\mu)\psi_n + \sum_{n,m=1}^{\infty}[\bar{B}^n_\mu, \bar{B}^m_\nu]\psi_n\psi_m \ , \qquad (66)
$$

$$
F_{z\mu} = i\partial_\mu \Pi\, \phi_0 + \sum_{n=1}^{\infty}\{iX^n\delta_{\mu 0} + \bar{B}^n_\mu(x^\mu)\}\dot{\psi}_n + i\sum_{n=1}^{\infty}[\Pi, \bar{B}^n_\mu]\psi_n\phi_0 \ . \qquad (67)
$$

Notice that $iX^n\delta_{\mu 0}$ does not contribute to $F_{\mu\nu}$. The additional terms due to $iX^n\delta_{\mu 0}$ come from $F_{\mu z}$. They contribute to the induced action through

$$
\begin{aligned}
F_{\mu\nu}F_{\mu\nu} &= (\partial_\mu \bar{B}^n_\nu - \partial_\nu \bar{B}^n_\mu)(\partial_\mu \bar{B}^m_\nu - \partial_\nu \bar{B}^m_\mu)\psi_n\psi_m \\
&\quad + \left\{[\bar{B}^n_\mu, \bar{B}^m_\nu] \ , (\partial_\mu \bar{B}^l_\nu - \partial_\nu \bar{B}^l_\mu)\right\}\psi_n\psi_m\psi_l \ , \qquad (68) \\
F_{\mu z}F_{\mu z} &= -(\partial_\mu \Pi \partial_\mu \Pi)\phi_0^2 + \bar{B}^n_\mu \bar{B}^m_\mu \dot{\psi}_n\dot{\psi}_m - X^n X^m \dot{\psi}_n\dot{\psi}_m \\
&\quad + \left(i\{\partial_\mu \Pi \ , \bar{B}^n_\mu\} - 2(\partial_0 \Pi)X^n\right)\phi_0\dot{\psi}_n + 2iX^n\bar{B}^m_0\dot{\psi}_n\dot{\psi}_m \\
&\quad + \left(-2X^n[\Pi \ , \bar{B}^m_0] + i\{\bar{B}^n_\mu \ , [\Pi \ , \bar{B}^m_\mu]\}\right)\dot{\psi}_n\psi_m\phi_0 + \left\{\partial_\mu \Pi \ , [\bar{B}^n_\mu \ , \Pi]\right\}{\phi_0}^2\psi_n \ , \qquad (69) \\
[F^4] &= (2\pi\alpha')^4\frac{9}{4}\frac{R^6}{U_{KK}U_z^2}\left\{i(\partial_0\Pi)\phi_0 + (iX^n + \bar{B}^n_0)\dot{\psi}_n\right\}\frac{1}{2}F_{ij}F_{ij}\left\{i(\partial_0\Pi)\phi_0 + (iX^m + \bar{B}^m_0)\dot{\psi}_m\right\} \\
[F^5] &\sim (2\pi\alpha')^5 X^n X^m \dot{\psi}_n\dot{\psi}_m\frac{1}{2}(F_{12}F_{23}F_{31} - F_{31}F_{12}F_{23}) \qquad (70)
\end{aligned}
$$

For notational simplicity we have omitted $\sum_{n=1}^{\infty}$.

The induced DBI action at finite density can be separated into field independent and dependent parts P_0 and P_1 respectively

$$
\begin{aligned}
P_0 &\equiv 1 - (2\pi\alpha')^2\frac{9}{4}\frac{U_z}{U_{\rm KK}}X^n X^m \dot{\psi}_n\dot{\psi}_m \\
P_1 &\equiv (2\pi\alpha')^2\frac{R^3}{2U_z^3}F_{\mu\nu}F_{\mu\nu} + (2\pi\alpha')^2\frac{9}{4}\frac{U_z}{U_{\rm KK}}(F_{\mu z}F_{\mu z} + X^n X^m \dot{\psi}_n\dot{\psi}_m) + [(F)^4] + [(F)^5] (71)
\end{aligned}
$$

so that

$$
\begin{aligned}
S_{DBI} &= \widetilde{T}\int d^4x dz\, U_z^2 \ \mathrm{tr}\ \sqrt{P_0+P_1} \\
&= \widetilde{T}\int d^4x dz\, U_z^2 \ \mathrm{tr}\ \left[\sqrt{P_0}+\frac{1}{2}\frac{P_1}{\sqrt{P_0}}-\frac{1}{8}\frac{P_1^2}{\sqrt{P_0}^3}+\cdots\right] \\
&= S_1(X^n)+S_2[\Pi, v_\mu; X^n]+\cdots\ ,
\end{aligned}
$$

with

$$
S_1(X^n) \equiv \widetilde{T}\int d^4x dz\, U_z^2\, \mathrm{tr}\ \sqrt{1-(2\pi\alpha')^2\frac{9}{4}\frac{U_z}{U_{\rm KK}}X^n X^m \dot{\psi}_n\dot{\psi}_m} \tag{72}
$$

$$
S_2[\Pi, v_\mu; X^n] \equiv \widetilde{T}\int d^4x dz\, U_z^2\, \mathrm{tr}\ \left[\frac{1}{2}\frac{P_1}{\sqrt{P_0}}-\frac{1}{8}\frac{P_1^2}{\sqrt{P_0}^3}+\cdots\right]\ . \tag{73}
$$

We note that the dependence on the chemical potential μ in S_1 is of the the type $\sqrt{1-\#\mu^2}$. There is no term linear in μ(except through the Chern-Simons term which is odd under $t\to -t$). The reason is that the QCD partition function is even under $\mu\to-\mu$ since the matter spectrum is symmetric around zero quark virtuality. We believe that this behavior maybe derived geometrically from a change in the underlying metric, but we are unable to show it.

The chemical potential μ as defined in (50) with (61)(i.e. $\mathcal{V}_\mu = -i\,\mu\,\delta_{\mu 0}\,1_{N_f\times N_f},\ \mathcal{A}_\mu=0$) yields $F_{0z}=0$. Thus there is no contribution to the DBI action. The only contribution stems from the Chern-Simons action as can be checked explicitly. However the definition (50) is at odd with VMD as explained in detail in [2]. In other words the physical vector field are $\tilde{v}_\mu^n$ defined in (52) and not v_μ^n in (50). So (50) has to be substituted by (62). In this case $F_{0z}\neq 0$ and μ contributes to the DBI action but not to the Chern-Simons action.

4 Free energy of dense matter

On the boundary the induced DBI action describes a dense hadronic system at finite baryon density whereby the baryons are Skyrmions in large N_c. The free energy Ω is given as

$$
\begin{aligned}
\Omega &\equiv -T\ln Z_G \\
&= -T\ln\left(N'\int \mathcal{D}[v_\mu]\mathcal{D}[\Pi]e^{-S^E_{DBI}}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= -kT\ln\left(N'\int \mathcal{D}[v_\mu]\mathcal{D}[\Pi]e^{-(S_1(X^1)+S_2[\Pi,v_\mu;X^n])}\right) \\
&= TS_1 - T\ln\left(\int \mathcal{D}[v_\mu]\mathcal{D}[\Pi]e^{-S_2[\Pi,v_\mu;X^n])}\right) \qquad (74)
\end{aligned}
$$

where Z_G is the grand partition function and N' is an irrelevant matter independent constant. The Gibbs relation

$$
\Omega = U - TS - \mu N = -PV \qquad (75)
$$

implies that $\Omega(T=0) = U - \mu N$ is the free energy at zero temperature and finite baryon density. Ignoring the meson zero-point contributions (classical limit), the leading contribution to the free energy per unit volume (minus the pressure) is

$$
\begin{aligned}
\epsilon(X^n) &\equiv \frac{\Omega(T=0)}{V_3} = \frac{S_1(X^n)-S_1(0)}{\beta V_3} \\
&= \widetilde{T}\int dz\, U_z^2\, \mathrm{tr}\left(\sqrt{1-(2\pi\alpha')^2\frac{9}{4}\frac{U_z}{U_{\mathrm{KK}}}X^nX^m\dot{\psi}_n\dot{\psi}_m}-1\right) \\
&= \frac{N_fN_cM_{KK}^4\lambda^3}{27^3\pi^5}\int dZ\, K^{\frac{2}{3}}\left(\sqrt{1-\frac{27^3\pi^5X^nX^m}{N_cM_{KK}^2\lambda^3}K^{\frac{1}{3}}\dot{\Psi}_n\dot{\Psi}_m}-1\right)\ , \qquad (76)
\end{aligned}
$$

where $K\equiv 1+Z^2$, $\lambda\equiv g_{YM}^2N_c$, and $\Psi_n \equiv \sqrt{\kappa}\psi_n(Z) = \sqrt{\tilde{T}(2\pi\alpha)^2R^3}\psi_n(Z)$.
In terms of the meson physical parameters fixed at zero density,[2]

$$
\begin{aligned}
g_{YM}^2N_c &= f_{\pi0}^2\frac{27\pi^4}{N_cM_{KK}^2} \\
M_{KK}^2 &= \frac{m_{k0}^2}{\lambda_k} \quad k=1,2,3,\cdots\ ,
\end{aligned}
$$

the free energy density becomes

$$
\epsilon = \frac{C_1N_ff_{\pi0}^6}{N_c^2m_{10}^2}\int dZ\, K^{\frac{2}{3}}\left(\sqrt{1-K^{\frac{1}{3}}\frac{m_{10}^4N_c^2}{D_1f_{\pi0}^6}\left(\alpha^n\mu-\widetilde{v}_{0C}^n\right)\left(\alpha^m\mu-\widetilde{v}_{0C}^m\right)\dot{\Psi}_{2n-1}\dot{\Psi}_{2m-1}}-1\right)
$$

with

$$
\begin{aligned}
\alpha^n &= \sqrt{\kappa}\int dZK^{-1/3}\Psi_{2n-1} \sim \sqrt{N_c}\ , \\
\mu &= \frac{\mu_B}{N_c}, \\
C_1 &\equiv \lambda_1\pi^7\ , \qquad D_1\equiv\lambda_1^2\pi^7\ . \qquad (77)
\end{aligned}
$$

[2]From this section the subscript 0 in the physical quantity, like $f_{\pi0}$, means the value at zero density.

The sum over the vector meson-species m, n is subsumed. (77) receives corrections from the quantum loops with mesonic insertions. In a way (77) captures the deformation of the underlying D8-probe-brane in the presence of a fixed External $\mathcal{V}_0$ to leading order. The square root shows that the response is not analytic, with a subtle branch point singularity to be physically interpreted below.

In the limit $N_c \gg 1$ but fixed baryon density $\mu_B = \mu N_c$ (low density limit),

$$\begin{aligned} & K^{\frac{1}{3}} \frac{m_{10}^4 N_c^2}{D_1 f_{\pi 0}^6} \left(\alpha^n \frac{\mu_B}{N_c} - \widetilde{v}_{0C}^n\right) \left(\alpha^m \frac{\mu_B}{N_c} - \widetilde{v}_{0C}^m\right) \dot{\Psi}_{2n-1} \dot{\Psi}_{2m-1} \\ \sim \; & \frac{1}{N_c} (\sqrt{N_c} \frac{1}{N_c} - \widetilde{v}_{0C}^n)^2 \sim \frac{1}{N_c^2} \end{aligned} \tag{78}$$

The free energy density can be Tayor expanded as

$$\begin{aligned} \epsilon & \approx -\frac{N_f M_{KK}^2}{2} \left(\alpha^n \frac{\mu_B}{N_c} - \widetilde{v}_{0C}^n\right) \left(\alpha^m \frac{\mu_B}{N_c} - \widetilde{v}_{0C}^m\right) \int dZ K \dot{\Psi}_{2n-1} \dot{\Psi}_{2m-1} \\ & \approx -\frac{N_f}{2} \sum_{n=1}^{\infty} m_{(2n-1)0}^2 (\sqrt{N_c} \frac{1}{N_c} - \widetilde{v}_{0C}^n)^2 \sim \frac{N_f}{N_c} \end{aligned}$$

which is quadratic in μ_B and of order N_f/N_c which is small.

In the limit $N_c \gg 1$ but fixed quark density $\mu = N_c/\mu_B$ (high density limit),

$$\begin{aligned} & K^{\frac{1}{3}} \frac{m_{10}^4 N_c^2}{D_1 f_{\pi 0}^6} (\alpha^n \mu - \widetilde{v}_{0C}^n) (\alpha^m \mu - \widetilde{v}_{0C}^m) \dot{\Psi}_{2n-1} \dot{\Psi}_{2m-1} \\ \sim \; & \frac{1}{N_c} (\sqrt{N_c} - \widetilde{v}_{0C}^n)(\sqrt{N_c} - \widetilde{v}_{0C}^m) \sim (N_c)^0 \end{aligned} \tag{79}$$

so that

$$\begin{aligned} \epsilon & = \frac{C_1 N_f f_{\pi 0}^6}{N_c^2 m_{10}^2} \int dZ \, K^{\frac{2}{3}} \left(\sqrt{1 - K^{\frac{1}{3}} \frac{m_{10}^4 N_c^2}{D_1 f_{\pi 0}^6} (\alpha^n \mu - \widetilde{v}_{0C}^n) (\alpha^m \mu - \widetilde{v}_{0C}^m) \dot{\Psi}_{2n-1} \dot{\Psi}_{2m-1}} - 1 \right) \\ & \sim -\frac{N_f}{2} \sum_{n=1}^{\infty} m_{(2n-1)0}^2 (X^n)^2 \sim N_f \, N_c \quad (X^n \ll 1) \end{aligned} \tag{80}$$

which is of order $N_f N_c$.

In Fig. 1 we show the free energy versus $X^1/m_{\rho 0}$ for $N_c = 3, N_f = 2$ and $m_{\rho 0} = 776\,\text{MeV}$, $f_{\pi 0} = 93$ MeV. Since [2]

$$\frac{m_{k0}^2}{M_{KK}^2} = \lambda_k = 0.67, \; 2.88, \; 6.6, \; 11.8, \; \cdots \tag{81}$$

only the $n = m = 1$ contribution is justified at low energy. So $X \equiv X^1$. Here X plays the role of a gauge-shifted chemical potential.

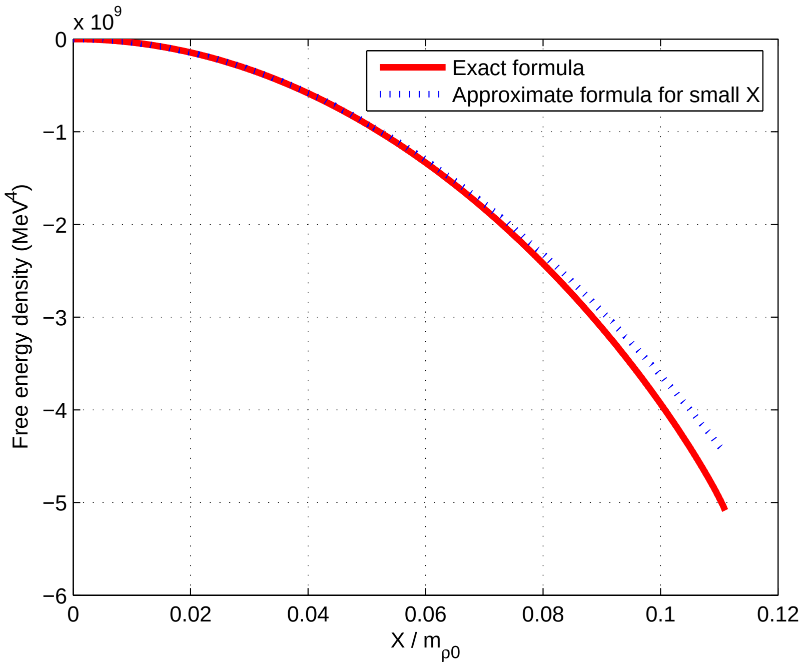

Figure 1: Free energy density vs $\frac{X}{m_{\rho 0}}$

The baryon density n_B follows from the free energy through

$$\begin{aligned} n_B \quad &:= \quad \frac{N_B}{V_3} = -\frac{1}{V_3}\frac{\partial\Omega}{\partial\mu} \\ &= \quad N_f M_{KK}^2 \int dZ K \frac{\alpha^1 X^1 \dot{\Psi}_1^2}{\sqrt{1 - K^{\frac{1}{3}} \frac{m_{v0}^4 N_c^2}{C_2 f_{\pi 0}^6} (X^1)^2 \dot{\Psi}_1^2}} \\ &\sim \quad \begin{cases} \int dZ \frac{\sqrt{N_c}\frac{1}{\sqrt{N_c}}}{\sqrt{1-1/N_c^2}} & \sim N_c^0 \quad \text{fixed } \mu_B \\ \int dZ \frac{\sqrt{N_c}\sqrt{N_c}}{\sqrt{1-(N_c)^0}} & \sim N_c \quad \text{fixed } \mu \end{cases} \end{aligned} \tag{82}$$

where again only the $n = m = 1$ contribution was retained. This is a transcendental equation that reexpresses the shifted chemical potential X in terms of the physical baryon density n_B. For large N_c and fixed μ_B it shows that $n_B \sim N_c^0$ which we interpret as the low density phase. For large N_c and fixed $\mu = \mu_B/N_c$ it shows that $n_B \sim N_c$ which we interpret as the high density phase. In Fig. 2 we show the numerical solution to (82) for n_B/n_0 versus $X/m_{\rho 0}$, where $n_0 = 0.17\,\mathrm{fm}^3$ is nuclear matter density. Asymptotically large densities are attained for a critical $X/m_{\rho 0} \approx 0.05\sqrt{D_1}\, f_{\pi 0}^3/N_c\, m_{\rho 0}^3$. For $N_c = 3$ and $N_f = 2$ we have $X/m_{\rho 0} \sim 0.111$ and $n_B/n_0 \sim 43$.

The existence of a limiting chemical potential reflects on the fact that the underlying D8-probe-brane becomes unstable if the external $\mathcal{V}_0$ becomes very large, as the square root in the ground state energy developes an imaginary part $i\pi$.

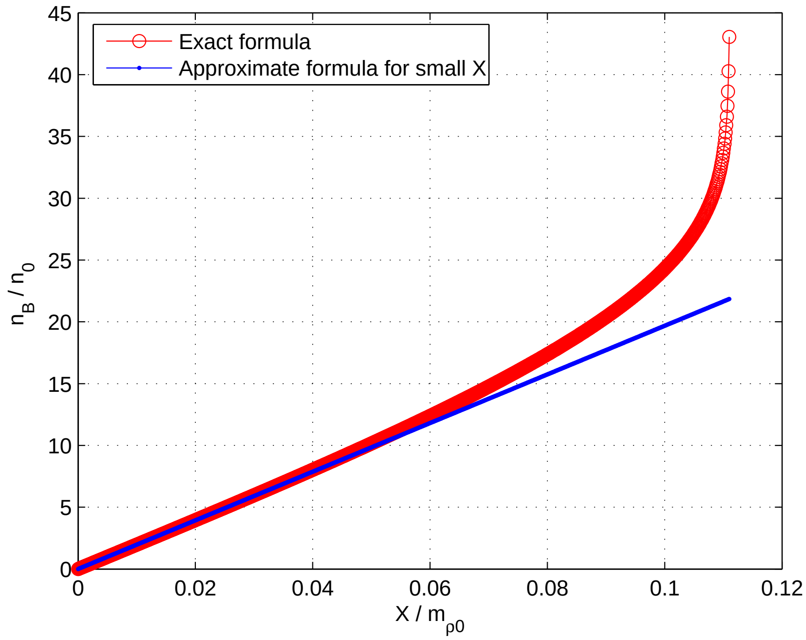

Figure 2: $\frac{n_B}{n_0}$ vs $\frac{X}{m_{\rho 0}}$

The pressure is $P = -\epsilon$ which is

$$P \;=\; \frac{C_1 N_f f_{\pi 0}^6}{N_c^2 m_{\rho 0}^2} \int dZ\, K^{\frac{2}{3}} \left(1 - \sqrt{1 - K^{\frac{1}{3}} \frac{m_{\rho 0}^4 N_c^2}{D_1 f_{\pi 0}^6} (X^1)^2 \dot{\Psi}_1^2} \right)$$

In Fig. 3 we show how the pressure changes with baryonic density n_B/n_0 for $N_c = 3$, $N_f = 2$, $m_{\rho 0} = 776$ MeV and $f_{\pi 0} = 93$ MeV. The pressure is dominated by *repulsive two-body* Skyrmion-Skyrmion like interactions at low density(the term of order n^2 in (83)). At high density *attractive three-body* Skyrmion-Skyrmion-Skyrmion like interactions appear and cause the pressure to saturate(the term of order n^3 in (83)). The two-body interaction are 100 times stronger throughout the density range explored. The specific fit is

$$P = (1.17\,10^7)\,n^2 - (2.14\,10^5)\,n^3 \tag{83}$$

If we recall that in matter the kinetic energy $K = (\partial P/\partial n)/2$, then the slope in Fig. 3 is a measure of the kinetic energy. The holographic principle suggests that hadronic matter at large N_c freezes at $n_B \sim 30\,n_0$.

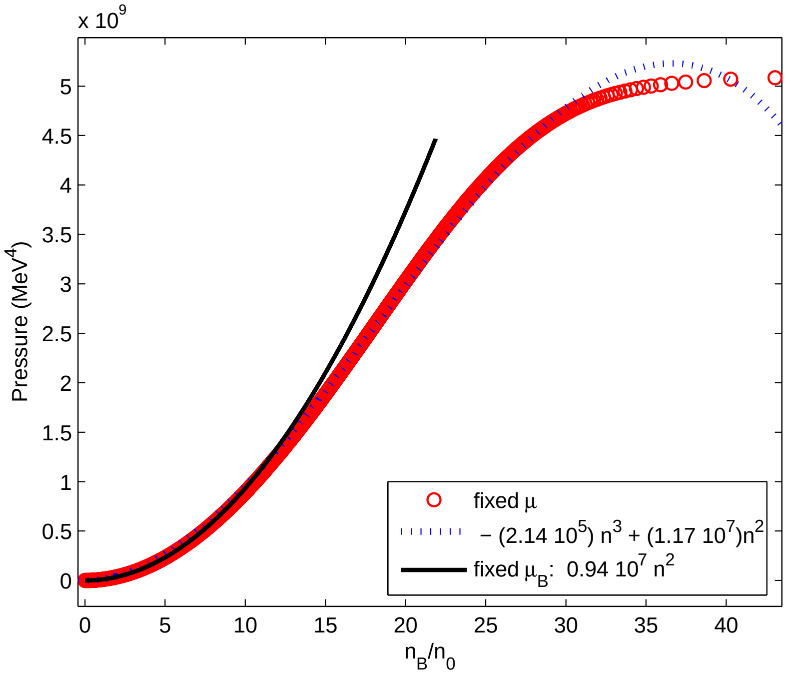

Figure 3: Pressure vs $\frac{n_B}{n_0}$

The energy density is defined as

$$\begin{aligned}
\mathcal{U} \;&:=\; \frac{U}{V_3} = \frac{\Omega + \mu N}{V_3} \\
&=\; \epsilon + \mu_B n_B = \epsilon + \frac{N_c}{\alpha^1}(\widetilde{v}^1_{0C} + X^1) n_B \\
&=\; \epsilon + \frac{N_c}{\alpha^1} X^1(n_B) n_B + \frac{N_c}{\alpha^1}\widetilde{v}^1_{0C} n_B
\end{aligned} \tag{84}$$

and is sensitive to the gauge choice of the *dynamical* $U(1)_v$ vector meson (omega). In the presence of $U(1)_V$ mesons, $\tilde{v}^n_\mu$, the energy density is dependent on the constant part of the $U(1)_V$ meson field, i.e. $\tilde{v}^n_{0C}$. This is reminicent of the gauge sensitivity of the EM potential. In the gauge where $\widetilde{v}^1_{0C} = 0$, the energy density is shown in Fig. 4. The repulsive three-body interactions soften the rise in the energy density. Note that the rise becomes almost linear after $n_B/n_0 \sim 30$. The slope is sensitive to the choice of the gauge $\widetilde{v}^1_{0C} = 0$. Since the pressure curve shows total freezing at these large densities, we expect that hadronic matter crystallizes. The slope is expected to be *negative* and a measure of *Madelung constant* for the specific crystal symmetry.

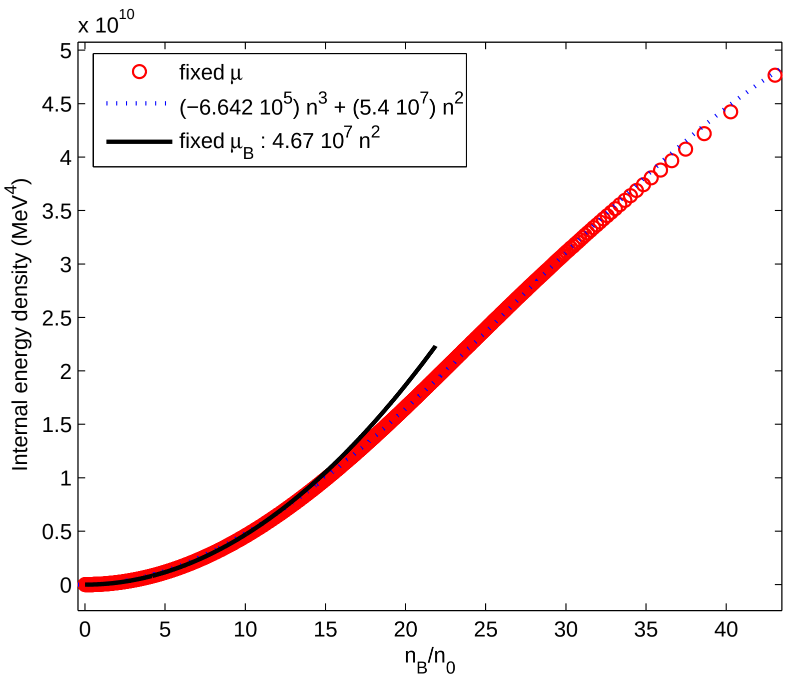

Figure 4: Internal energy density vs $\frac{n_B}{n_0}$

5 Mesons in Holographic Dense Matter

The DBI effective action in matter fixes completly the meson dispersion laws and interactions. Indeed, in matter $S_2[v_\mu; X^n]$ is

$$S_2[\Pi, v_\mu; X^n] \;=\; \int d^4x dZ \;\mathrm{tr}\left[Q\widetilde{T}U_{KK}U_z^2 P_1 - \frac{1}{4}Q^3\widetilde{T}U_{KK}U_z^2 P_1^2 + \cdots\right] . \tag{85}$$

where

$$Q(Z; X^n) \;:=\; \frac{1}{\sqrt{P_0}} = \frac{1}{\sqrt{1 - \Delta K^{\frac{1}{3}}\dot{\Psi}_1^2}}$$

$$\Delta := \frac{m_{v0}^4 N_c^2 (X^1)^2}{C_2 f_{\Pi 0}^6} = \frac{27^3 \pi^5 (X^1)^2}{N_c M_{KK}^2 \lambda^3} \quad \sim \frac{1}{N_c} \;\; \text{fixed} \;\; \mu_B \quad (\sim 1 \;\; \text{fixed} \;\; \mu) \tag{86}$$

Only the term $n = m = 1$ in the summation over the vector meson species was retained (See (81)). In this section we treat A_μ as anti-Hermitiaon matrices with generators t^a normalized as $\mathrm{tr}(t^a t^b) = \frac{1}{2}\delta_{ab}$. The effective action is rotated back to Minkowski space. Throughout, only terms up to the third order in the fields are retained , i.e.$\mathcal{O}((\Pi, v_\mu)^4)$. The leading terms to this order in $[F^5]$ vanishes because of the cyclicity of the trace.

The expansion can be simplified by using the notations

$$\bar{v}_0 := \overline{v_{0C}^1}, \quad X := X^1 \;, \quad v_i := v_i^1 \tag{87}$$

so that

$$\begin{aligned}
S_2[\Pi, v_\mu; \mu] \;\; = \;\; & \int d^4x \Bigg[a_{\Pi^2}^T \,\mathrm{tr}\,(\partial_0 \Pi \partial_0 \Pi) - a_{\Pi^2}^S \,\mathrm{tr}\,(\partial_i \Pi \partial_i \Pi) \\
& - a_{v^2}^T \,\mathrm{tr}(\partial_0 v_i - \partial_i \bar{v}_0)^2 + \frac{1}{2} a_{v^2}^S \,\mathrm{tr}(\partial_i v_j - \partial_j v_i)^2 \\
& - m_v^{2T} \,\mathrm{tr}\, \bar{v}_0^2 + m_v^{2S} \,\mathrm{tr}\, v_i^2 \\
& - a_{v^3}^T \,\mathrm{tr} \Big([\bar{v}_0, v_i](\partial_0 v_i - \partial_i \bar{v}_0) \Big) + a_{v^3}^S \,\mathrm{tr} \Big([v_i, v_j](\partial_i v_j - \partial_j v_i) \Big) \\
& - a_{v\Pi^2}^T \,\mathrm{tr} \Big(\partial_0 \Pi [\bar{v}_0, \Pi] \Big) + a_{v\Pi^2}^S \,\mathrm{tr} \Big(\partial_i \Pi [v_i, \Pi] \Big) \\
& - a_{\bar{v}_0 v^2}^T \,\mathrm{tr} \Big(\bar{v}_0 (\partial_0 v_i - \partial_i \bar{v}_0)^2 \Big) + a_{\bar{v}_0 v^2}^S \,\mathrm{tr} \Big(\bar{v}_0 (\partial_i v_j - \partial_j v_i)^2 \Big) \\
& - a_{\bar{v}_0 \Pi^2} (\partial_0 \Pi \partial_0 \Pi \bar{v}_0 - \partial_0 \Pi \partial_i \Pi v_i) - a_{v_0 v^2} \bar{v}_0 (\bar{v}_0^2 - v_i^2) \\
& + \left(\int dZ K Q \phi_0 \dot{\Psi}_1 \right) \mathrm{tr} \left(\frac{2i}{f_\Pi} (\partial_\mu \Pi) v_\mu + \mu \frac{2i}{f_\Pi} (\partial_0 \Pi) + \mu [\Pi]^3 \right) \\
& + \left\{ \sim \mu (\partial_0 \Pi)(\partial_i v_j - \partial_j v_i) \int dZ \phi_0 \dot{\Psi}_1 \Psi_1^2 \right\} \Bigg] \;.
\end{aligned} \tag{88}$$

The last two contributions in (88) vanish because K, Q, Ψ_1 and ϕ_0 are even functions and $\dot{\Psi}_1$ is an odd function. The new two terms related to $\bar{v}_0$ are in the 6th and 7th lines of (88). All the parameters appearing in (88) are tabulated below for non-zero density. Their respective values at zero density are recorded on the right-most column for comparison. We note that $Q, \Delta \sim N_c^0$ for fixed μ, so that all coefficients are of order N_c^0 for fixed μ at large N_c.

Coefficients	nonzero X	$X=0(Q=1,\Delta=0)$
$a^T_{\Pi^2}$	$\frac{1}{\pi}\int dZ K^{-1}\, Q\left(1+\Delta Q^2 K^{\frac{1}{3}}\dot\Psi_1^2\right)$	1
$a^S_{\Pi^2}$	$\frac{1}{\pi}\int dZ K^{-1}\, Q$	1
$a^T_{v^2}$	$\int dZ\, K^{-\frac{1}{3}}\Psi_1^2\, Q$	1
$a^S_{v^2}$	$\int dZ\, K^{-\frac{1}{3}}\Psi_1^2\, Q^{-1}$	1
$m_v^{2\,T}$	$\frac{m_{v0}^2}{\lambda_1}\int dZ\, K\dot\Psi_1^2\, Q\left(1+\Delta Q^2 K^{\frac{1}{3}}\dot\Psi_1^2\right)$	m_ρ^2
$m_v^{2\,S}$	$\frac{m_{v0}^2}{\lambda_1}\int dZ\, K\dot\Psi_1^2\, Q$	m_ρ^2
$a^T_{v^3}$	$\frac{2m_{v0}}{\sqrt{\pi\lambda_1}f_{\Pi 0}}\int dZ\, K^{-\frac{1}{3}}\Psi_1^3\, Q$	$\frac{2m_{v0}}{\sqrt{\pi\lambda_1}f_{\Pi 0}}\cdot 0.45$
$a^S_{v^3}$	$\frac{2m_{v0}}{\sqrt{\pi\lambda_1}f_{\Pi 0}}\int dZ\, K^{-\frac{1}{3}}\Psi_1^3\, Q^{-1}$	$\frac{2m_{v0}}{\sqrt{\pi\lambda_1}f_{\Pi 0}}\cdot 0.45$
$a^T_{v\Pi^2}$	$\frac{2}{\sqrt{\kappa\pi}}\int dZ\, K^{-1}\Psi_1^2\, Q\left(1+\Delta Q^2 K^{\frac{1}{3}}\dot\Psi_1^2\right)$	$\frac{2}{\sqrt{\kappa\pi}}\cdot 0.63$
$a^S_{v\Pi^2}$	$\frac{2}{\sqrt{\kappa\pi}}\int dZ\, K^{-1}\Psi_1^2\, Q$	$\frac{2}{\sqrt{\kappa\pi}}\cdot 0.63$
$a^T_{v_0 v^2}$	$\frac{\Delta}{2X}\int dZ\, Q^3\dot\Psi_1^2\Psi_1^2$	0
$a^S_{v_0 v^2}$	$-\Delta X\int dZ\, Q\dot\Psi_1^2\Psi_1^2+\frac{\Delta}{2X}\int dZ\, Q^3\dot\Psi_1^2\Psi_1^2+\frac{\Delta^2}{2X}\int dZ\, Q^3 K^{\frac{1}{3}}\dot\Psi_1^4\Psi_1^2$	0
$a_{v_0\Pi^2}$	$\frac{\Delta}{X}\int dZ\, Q^3 K^{-\frac{2}{3}}\dot\Psi_1^2$	0
$a_{v_0 v^2}$	$\frac{\Delta}{X}\int dZ\, Q^3 K^{\frac{4}{3}}\dot\Psi_1^4$	0

5.1 Meson Velocities

For $N_c=3$, $m_{vo}=776\text{MeV}$, $f_{\pi 0}=93\text{MeV}$, we show the pion in Fig. 5 and vector meson in Fig. 6 renormalization constants, both for the time and space-components. The resulting velocity of the pion and the rho meson in holographic dense matter is shown in Fig. 7. The pion velocity

$$v_\pi=\sqrt{\frac{a^S_{\pi^2}}{a^T_{\pi^2}}}=\frac{f^S_\pi}{f^T_\pi} \tag{89}$$

with $f_\pi^{S,T}$, the spatial (S) and temporal (T) pion decay constants, approaches zero when hadronic matter freezes. The vector meson velocity

$$v_v=\sqrt{\frac{a^S_{v^2}}{a^T_{v^2}}} \tag{90}$$

is about half.

5.2 Vector Screening Masses

The pion is massless throughout as the current quark masses are set to zero in this work. The vector meson masses follow by Higgsing, and change in matter both space-like and

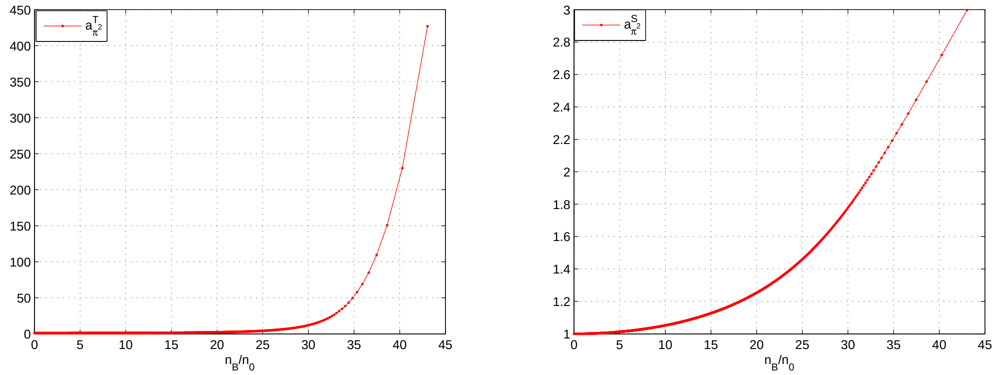

Figure 5: $a^T_{\pi^2}$ and $a^S_{\pi^2}$

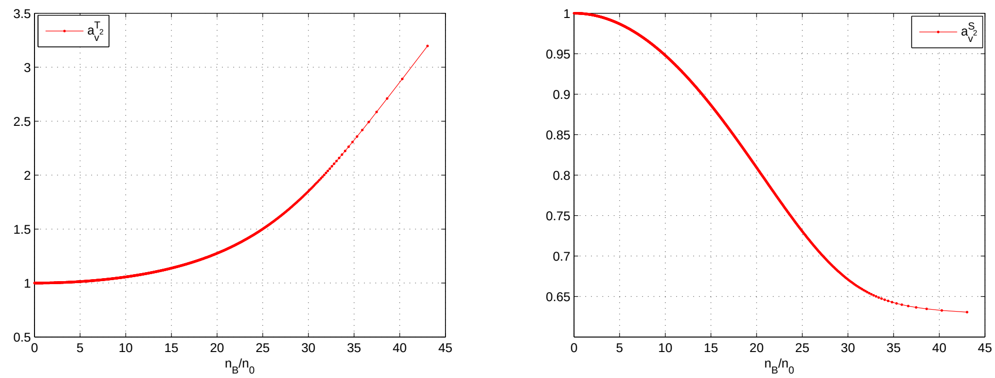

Figure 6: $a^T_{v^2}$ and $a^S_{v^2}$

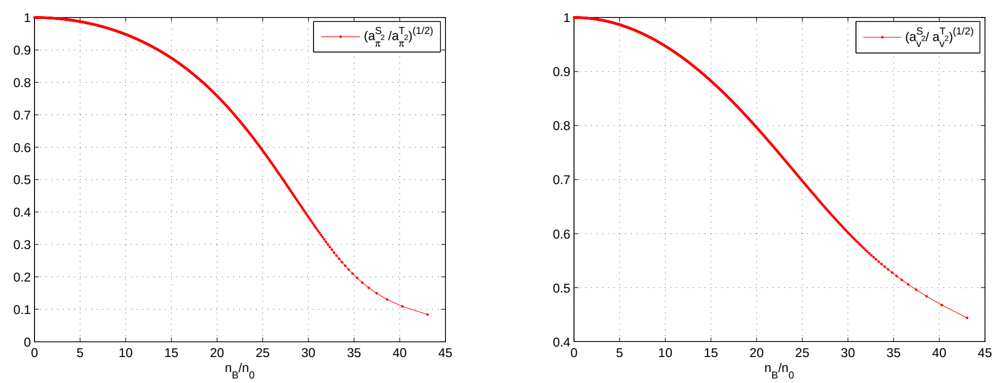

Figure 7: velocity: pion(left) and ρ(right)

time-like. Space-like, the vector meson masses are screening masses. The unrenormalized masses increase with increasing baryon density as shown in Fig. 8. The physical screening masses(M_T) are obtained by normalizing with the pertinent matter dependent *space-like* wavefunction renormalizations. Both the vector and isovector screening masses increase with increasing baryon density as they should. Fig. 9 show the screening masses for longitudinal (left) and transverse (right) vectors. They are defined as

$$M_L^2 = \frac{m_v^{2T}/m_{v0}^2}{a_{v^2}^T} \ , \qquad M_T^2 = \frac{m_v^{2S}/m_{v0}^2}{a_{v^2}^S} \ . \tag{91}$$

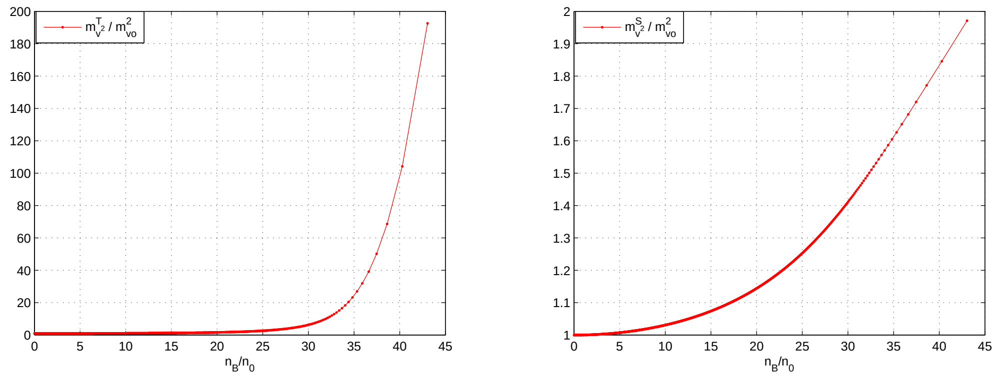

Figure 8: $\frac{m_v^{2\,T}}{m_{v0}^2}$ and $\frac{m_v^{2\,S}}{m_{v0}^2}$

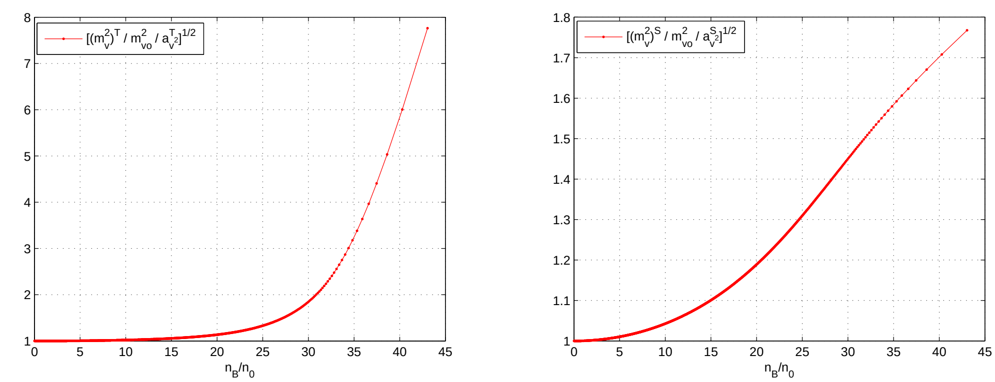

Figure 9: Screening masses: Longitudinal (left) and Transverse (right)

5.3 Vector Meson Masses

The time-like vector masses are obtained similarly to the screening masses by instead using the *time-like* wavefunction renormalizations. That is

$$M_V^2 = \frac{m_v^{2S}/m_{v0}^2}{a_{v^2}^T} \tag{92}$$

The dependence of M_V^2 on the baryon density is shown in Fig. 10. The rho meson mass drops by 20% when holographic dense matter freezes.

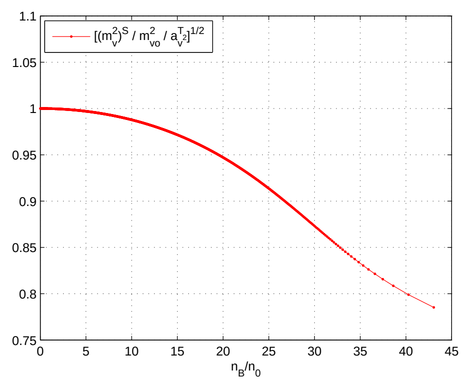

Figure 10: Time-like Vector mass vs baryon density

5.4 $V\pi\pi$ and VVV Couplings in Matter

The $V\pi\pi$ coupling is modified in matter. The longitudinal and transverse couplings are shown in Fig. 11 (left). The longitudinal coupling drops by 40% at the freezing point, while the transverse coupling *vanishes* when dense matter freezes. In Fig. 11 (right) VVV coupling also decrease in matter. The transverse couplings drop by 90%, while the longitudinal ones by 30%. In addition to these standard couplings, new matter dependent couplings emerge. Their composition and dependence on the baryon density is shown in Fig. 12.

Many of the results presented here in matter bears similarities (and differences) with arguments presented at finite temperature using phenomenological models with hidden local symmetry [8].

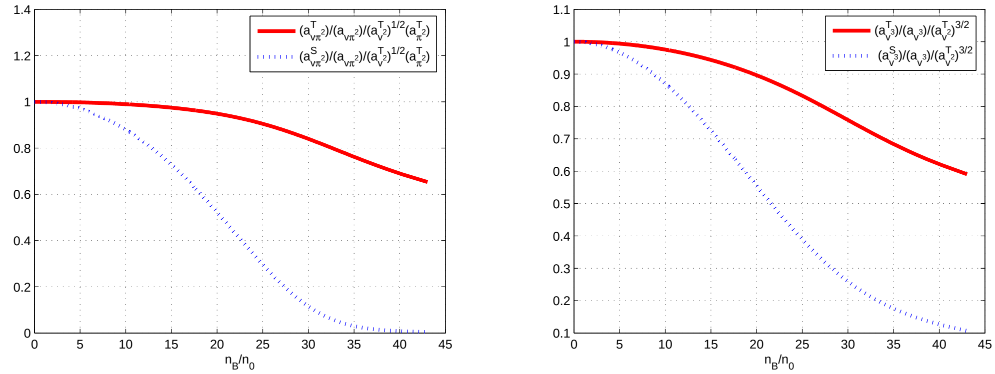

Figure 11: $V\pi\pi$ and VVV couplings.

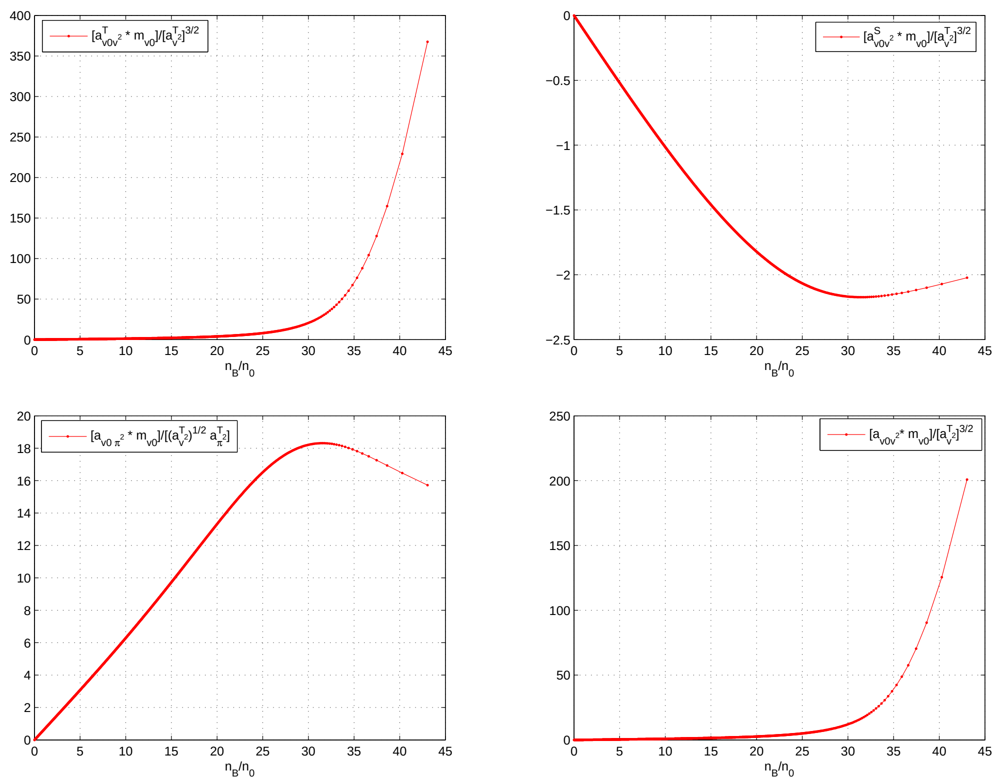

Figure 12: New terms

6 Conclusions

We have provided a minimal extension of holographic QCD [2] to dense matter. The induced DBI effective action on D8 in the presence of a constant external $U(1)_v$ source $\mathcal{V}_0$ is the effective action we expect from the gravity dual theory from general principles. Indeed, if we were to solve for the effect of the boundary $\mathcal{V}_0$ on the D4-brane background and the resulting DBI action on the D8-probe brane, the answer is the *externally gauged* DBI induced action in the vacuum. The effective action develops an imaginary time at large $\mathcal{V}_0$, a signal that the D8-probe-brane craks in the external field.

Holographic matter describes dense QCD at large N_c with baryons as solitons. Its bulk pressure asymptotes a constant at large density, signalling total freezing with zero mean kinetic energy. Before freezing the matter is dominated by two-body repulsion at low density and three-body attraction at intermediate densities. The two-body effects are 100 times stronger than the three-body effects.

In holographic matter the pions stall to almost a stop, while the vector mesons only slow down. The vector masses drop by about 20%, while vector screening becomes increasingly large. The transverse vector mesons completly decouple from pions at large densities. Many of the current results bear similarities with known results from effective models at large N_c. In a way, they are new as they provide first principle calculations to large N_c and strongly coupled QCD.

In the work to follow, we will present results for holographic matter including finite quark masses and temperature.

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